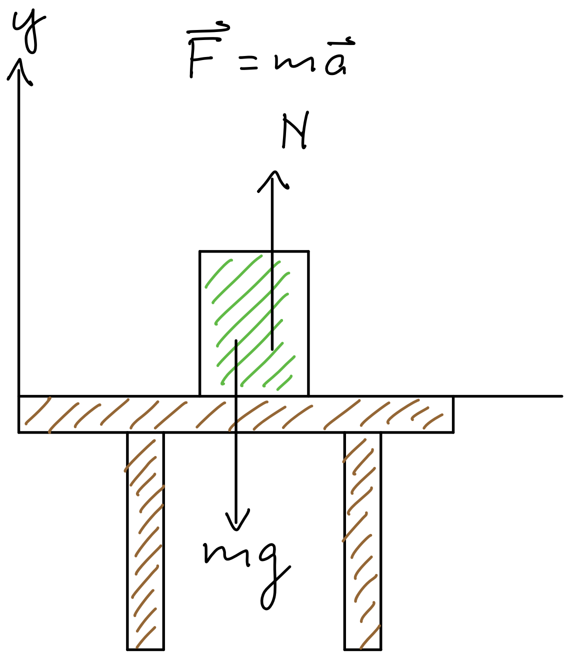
Example 5 (normal force and friction):

i) Consider a mass m sitting on a table as shown below:

$\vec{F} = m\vec{a}$



The force is a vector $\rightarrow$ $F_x = m a_x$

$F_y = m a_y$

- x-direction:

  $F_x = 0, \; a_x = 0 \Rightarrow 0 = 0$

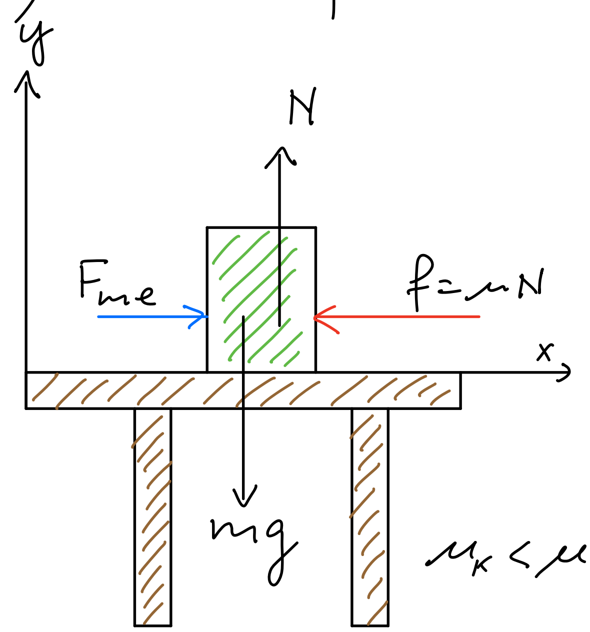
- y-direction:

  $N - mg = m a_y$

  know $a_y = 0$

  $\Rightarrow \quad N = mg$

ii) with "friction":



$\mu_k < \mu_s$

- For $F_{me} < \mu_s N$, (μ_s: coefficient of static friction)

  block doesn't move

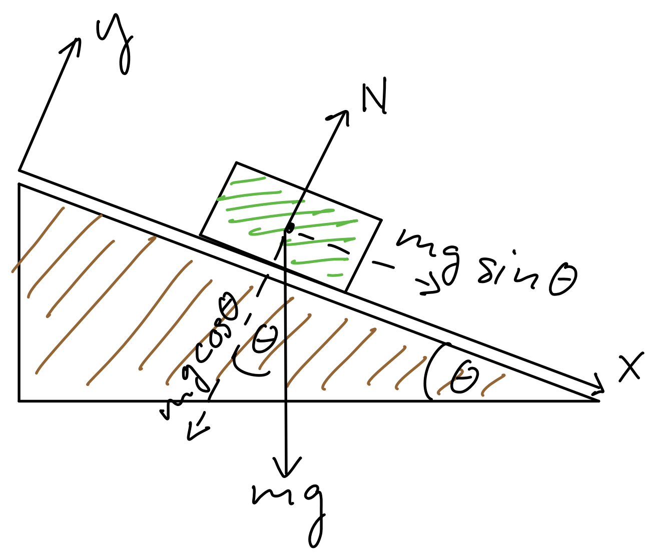
- For $F_{me} > \mu_s N$, block starts moving:

  $F_{me} - \mu_k N = m a_x$ (μ_k: coefficient of kinetic friction)

Example 6 (Inclined plane):

i) without friction:

Consider a mass m sitting on an inclined plane.



Then we obtain the following equations:

$$mg\sin\theta = ma_x \qquad (1)$$

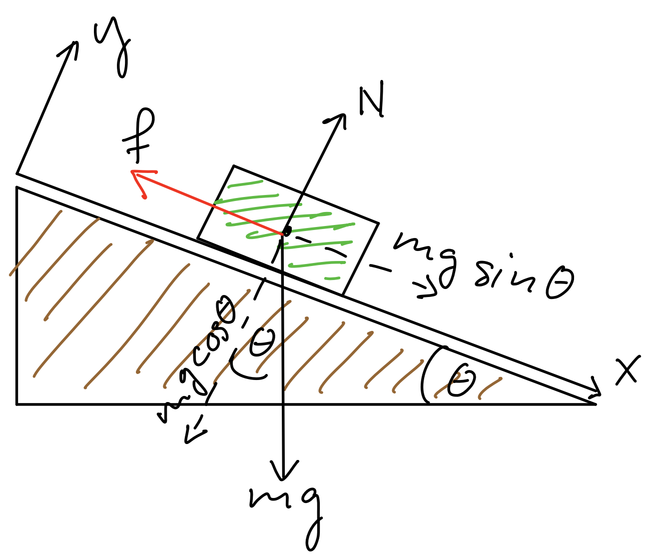
$$N - mg\cos\theta = ma_y \qquad (2)$$

We know $a_y = 0 \Rightarrow N = mg\cos\theta$

$(1) \Rightarrow a_x = g\sin\theta$

Thus, by making θ small, we can reduce the acceleration by factor $\sin\theta$

ii) with friction:



Relevant equations:

$$N - mg\cos\theta = ma_y \quad (1)$$

$$mg\sin\theta - f = ma_x \quad (2)$$

(1) + $a_y = 0 \Rightarrow N = mg\cos\theta$

Note: the static frictional force is not always $\mu_s N$; it is whatever it takes to keep the block still, up to a maximum of $\mu_s N$.

$\Rightarrow$ for $mg\sin\theta < \mu_s N$ we have

(2) $\Rightarrow$ $f = mg\sin\theta \quad (a_x = 0)$

If we increase θ up to an angle θ^* with $mg\sin\theta^* = \mu_s N = \mu_s mg\cos\theta^*$ then we can measure μ_s:

$$\mu_s = \tan\theta^*$$

Note: the mass has cancelled. So it doesn't matter on which planet we perform this experiment, the result will always be the same.

For $\tan\theta > \mu_s$ the block will begin to slide down $\rightarrow$ frictional force becomes $f_K = \mu_K N$ $(\mu_K < \mu_s)$

Relevant equations are:

$$N - mg\cos\theta = ma_y \quad (1) \quad \Rightarrow N = mg\cos\theta$$

$$mg\sin\theta - \mu_K N = ma_x \quad (2)$$

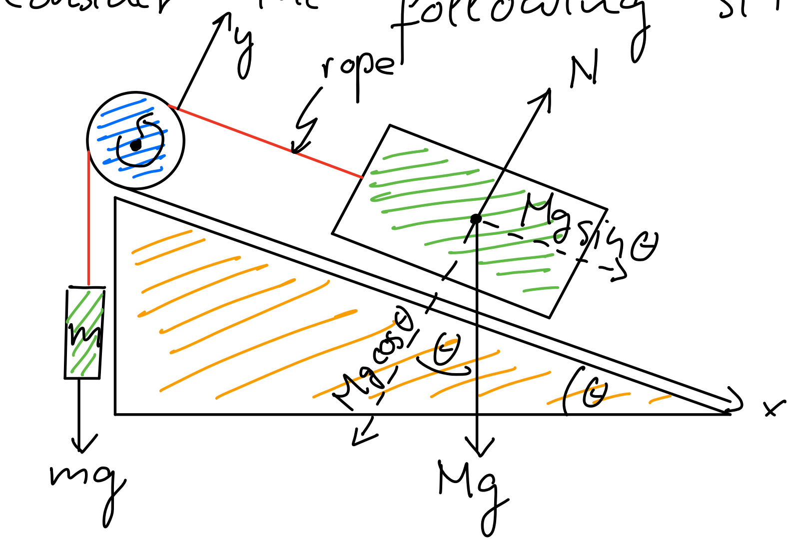
$\Rightarrow$ eq. (2) becomes:

$$mg\sin\theta - \mu_K mg\cos\theta = ma_x$$

$$\Leftrightarrow \quad g(\sin\theta - \mu_K\cos\theta) = a_x$$

Example 7 (coupled masses):

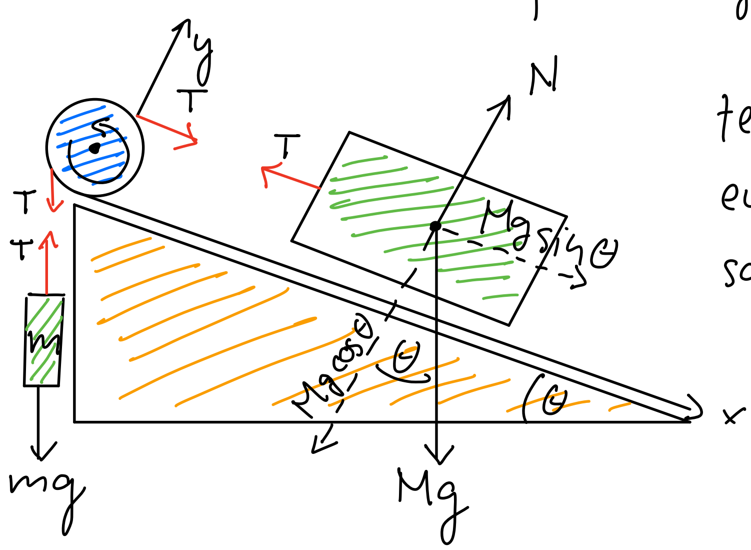
Consider the following situation



assume $M > m$

- the bigger mass M will go down the slope as $M \longrightarrow \infty$
- the smaller mass m will go straight down as $\theta \longrightarrow 0$

Let us draw the free-body diagram:



tension T is everywhere the same along the rope

$\rightarrow$ equation for M along x- axis:

$$Mg\sin\theta - T = Ma \qquad (1)$$

equation fom m is:

$$T - mg = ma \qquad (2)$$

adding (1) and (2) gives:

$$Mg\sin\theta - mg = Ma + ma$$

$$\Leftrightarrow \quad a = g\left[\frac{M\sin\theta - m}{m + M}\right]$$

$\rightarrow$ for a to be positive, we need

$$M\sin\theta > m$$

For negative $a < 0$, the same formula can be equivalently applied

Adding friction, we get

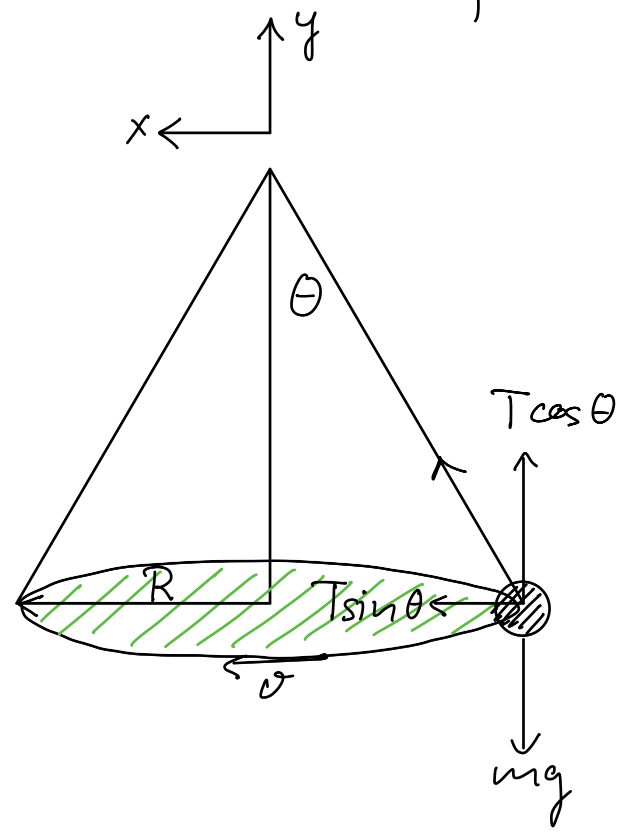
$$Mg\sin\theta - T - f = Ma$$

$$T - mg = ma$$

$$\Rightarrow \quad a = g\left[\frac{M\sin\theta - \overset{\mu M\cos\theta}{\frac{f}{g}} - m}{m + 1}\right]$$

§3.4 Circular motion

Consider the following situation:



Recall "centripetal" acceleration for circular motion:

$$a = |\vec{a}| = \omega^2 R$$

where ω is "angular velocity" given by $\omega = \frac{v}{R}$ "tangential velocity"

$$\Rightarrow a = \frac{v^2}{R}$$

→ relevant equations:

$$T\cos\theta - mg = 0 \quad \text{vertical}$$

$$T\sin\theta = ma = m\frac{v^2}{R} \quad \text{horizontal (radial)}$$

$$\Rightarrow \tan\theta = \frac{v^2}{Rg}$$

then $1 + \tan^2\theta = \frac{1}{\cos^2\theta}$

$$\rightarrow T = mg\sqrt{1 + \left(\frac{v^2}{Rg}\right)^2}$$

Example 8:

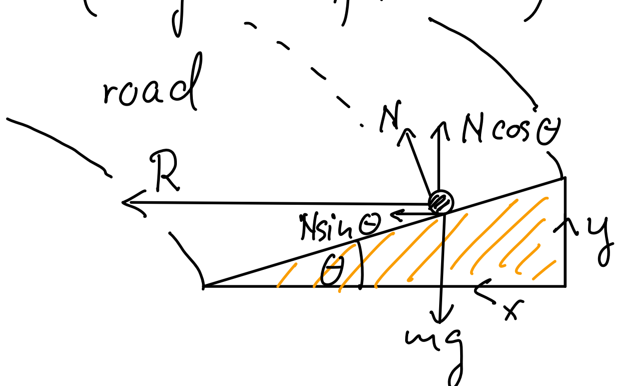
Imagine you are driving on a circular racetrack of radius R at speed v

→ need a force $m\frac{v^2}{R}$ on the car to bend it into a circle!

two solutions:

a) use friction $f \leq \mu_s mg$

b) tilt the road by an angle θ (neglect friction) as follows:



→ relevant equations:

$$\left.\begin{aligned} N\cos\theta &= mg \\ N\sin\theta &= m\frac{v^2}{R} \end{aligned}\right\} \Rightarrow \tan\theta = \frac{v^2}{Rg}$$

§4. Law of Conservation of Energy

§4.1 Introduction to energy

Consider a force that produces an acceleration $a = \frac{F}{m}$

in §1 we saw that for constant acceleration: $v^2 = v_0^2 + 2a(x - x_0)$

$$\longrightarrow v_2^2 = v_1^2 + 2\frac{F}{m}d$$

where $d = x - x_0 =: x_2 - x_1$

$$\Leftrightarrow \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = Fd \qquad (*)$$

<u>Definition 1:</u>

i) The combination $\frac{1}{2}mv^2$ is called "Kinetic energy", denoted by E_K

ii) The product Fd is called "work done by the force" and is denoted by W. units: $Nm =: J$ (Joules)

Equation (*) gives the

<u>Theorem 1</u> (work-energy theorem):

$$K_2 - K_1 = Fd = W$$

"The change in kinetic energy is equal to the work done by all the forces."

Next, imagine the work is done in a time interval Δt, distance $d = \Delta x$

$$\Rightarrow \lim_{\Delta t \to 0} \frac{\Delta K}{\Delta t} = \frac{dK}{dt} = F \frac{dx}{dt} = Fv =: P$$

<u>Definition 2:</u>

The quantity $P = Fv = \frac{dK}{dt}$ is defined as "power"

"rate at which work is done"

<u>Units</u>: $\frac{J}{s}$ (joules per second)

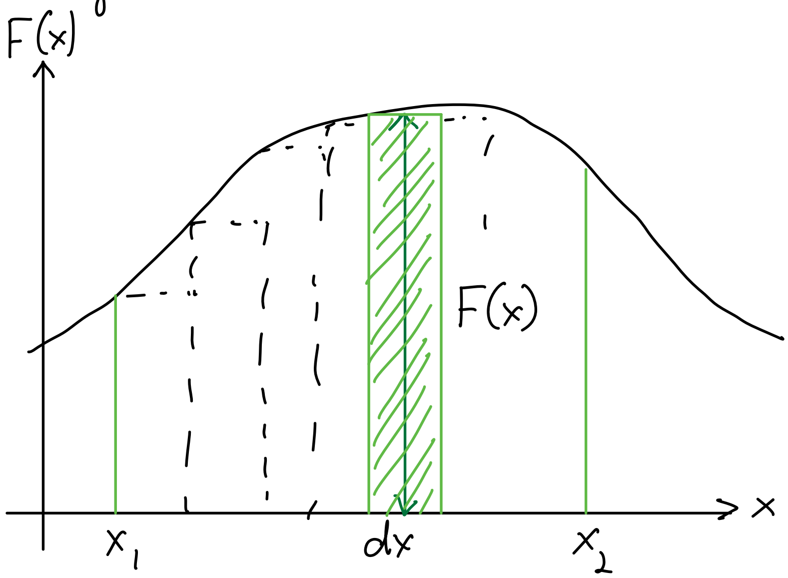
or W (watts)

→ a 60 W light bulb consumes energy at the rate 60 joules per second

Let us now look at forces that vary in time:

- force of a spring $F(x) = -kx$
- force of gravity (it is only locally approximately constant $= mg$)

In general:



→ cannot apply formula $W = Fd$ as F varies over distance d

→ for a small interval:

$dK = F(x)\,dx$ (area of thin rectangle)

→ have to sum up al small rectangles below graph

→ going from x_1 to x_2 means computing area under graph "Riemann integral"

→ work-energy theorem in general form:

$$K_2 - K_1 = \int_{x_1}^{x_2} F(x)\,dx =: W$$

Use "Fundamental Theorem" of calculus to compute it:

- find function $G(x)$ with

$$F(x) = \frac{dG(x)}{dx}$$

- then $\int_{x_1}^{x_2} F(x)\,dx = G(x_2) - G(x_1)$

$$\int_{x_1}^{x_2} F(x)\,dx = \lim_{n\to\infty} \sum_{l=1}^{n} \overbrace{\frac{G\left(x_1 + l\,\frac{x_2-x_1}{n}\right) - G\left(x_1 + (l-1)\frac{x_2-x_1}{n}\right)}{\frac{1}{n}}}^{\approx F\left(x_1 + l\,\frac{x_2-x_1}{n}\right)} \cdot \frac{1}{n}$$

$$= G(x_2) - G(x_1)$$